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MODEL UNCERTAINTY IN SEQUENTIAL DECISION MAKING

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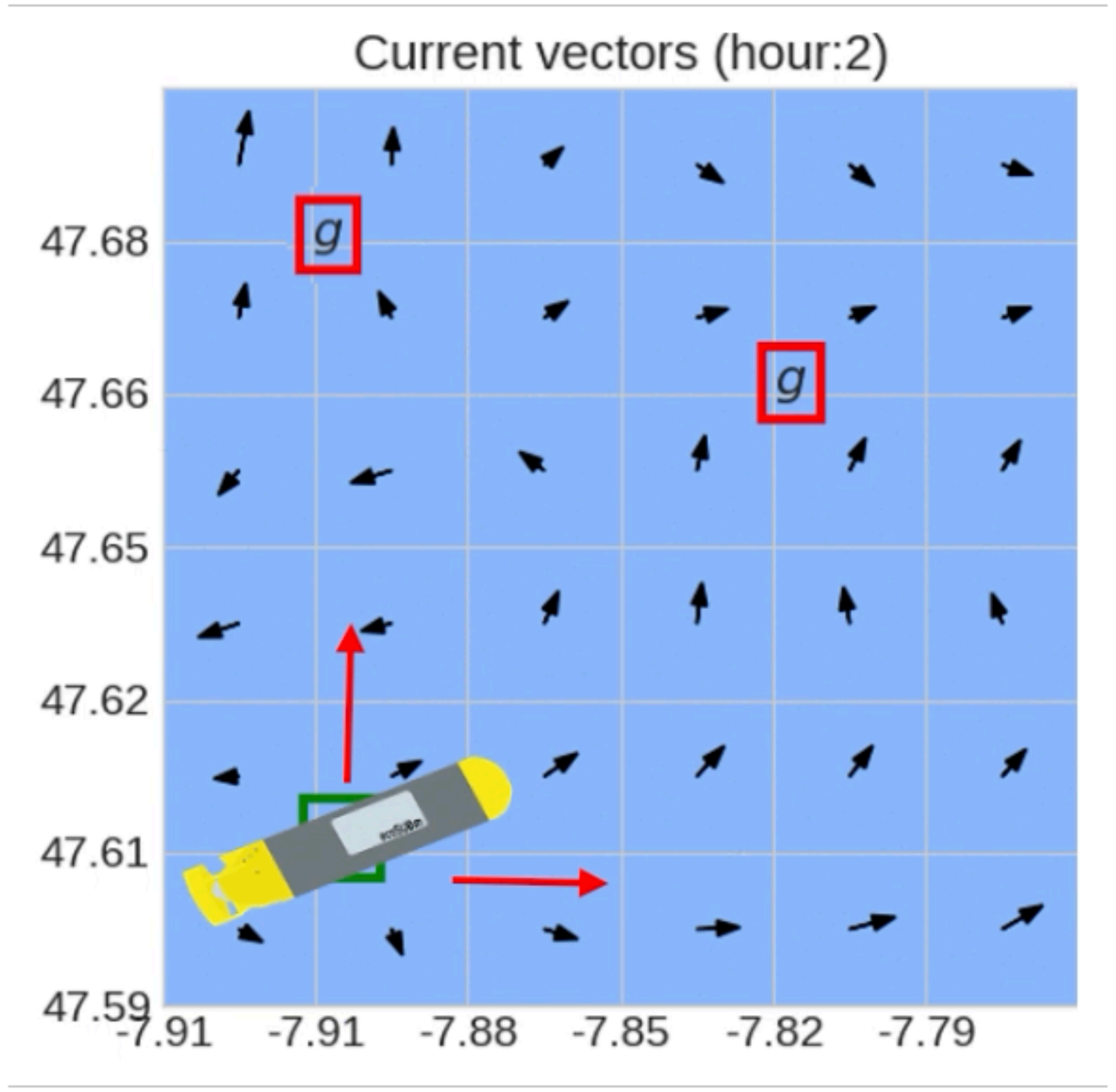
Introduction

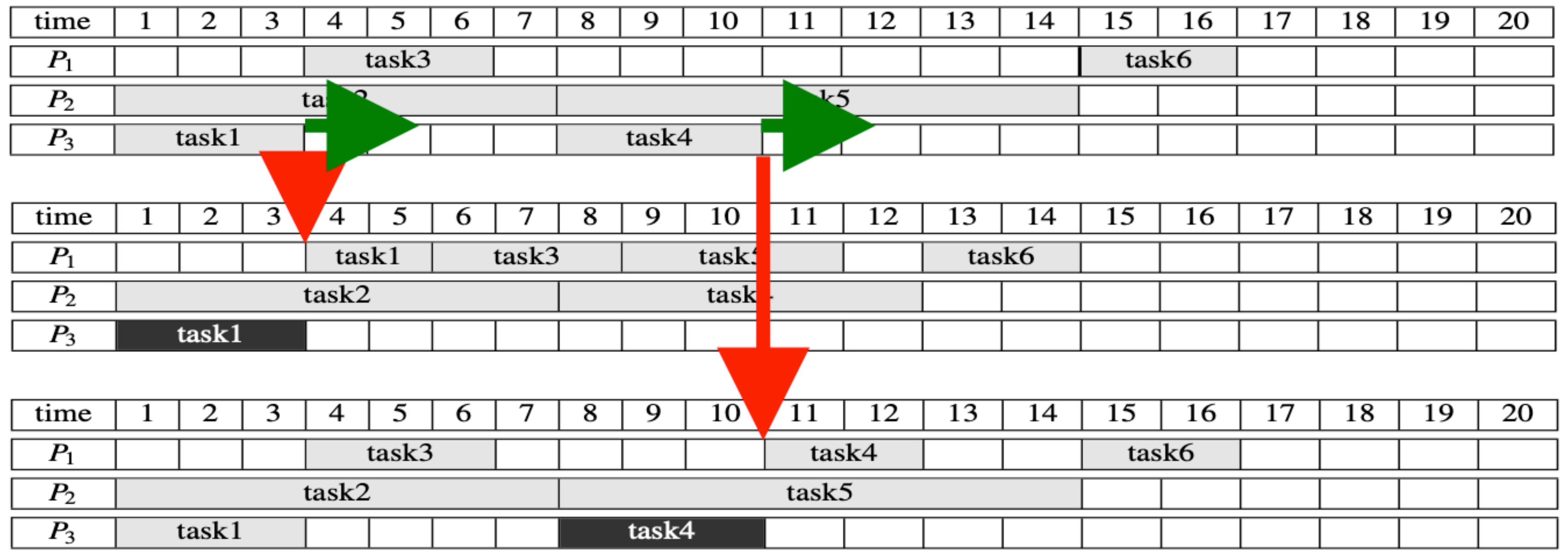
Sequential decision making under uncertainty

- Sequential decision making
  - iterative interaction with an environment to achieve a goal
  - sequential process of making observations and executing actions
  - applications in: health, energy, transportation, robotics, …



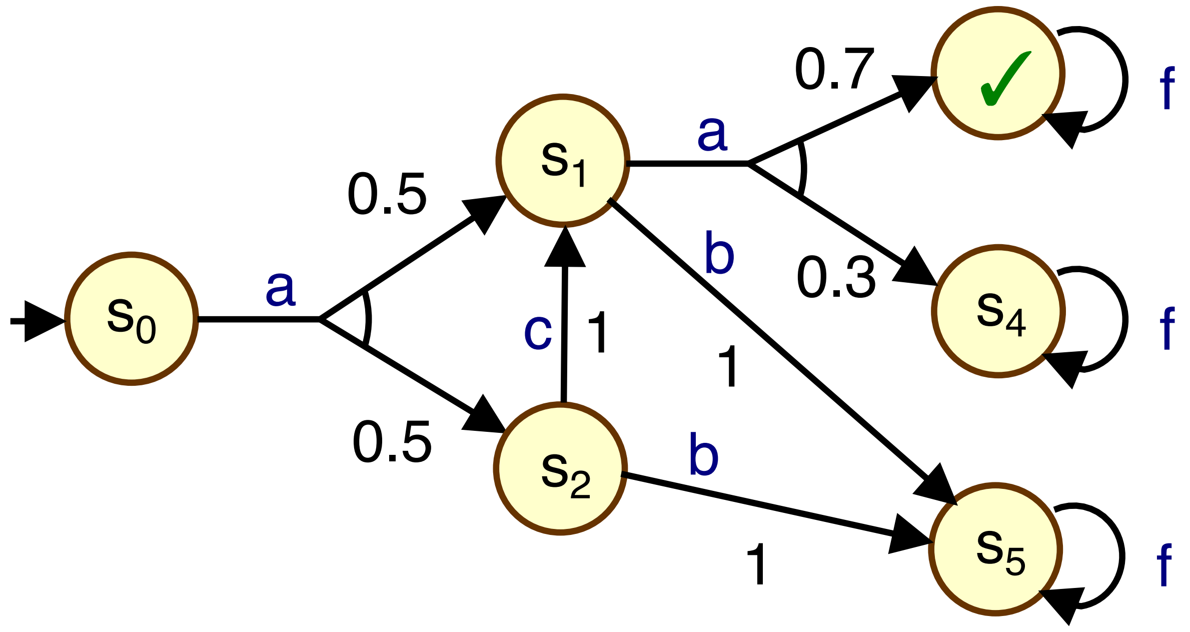
- Sequential decision making under uncertainty
  - noisy sensors, unpredictable conditions, lossy communication, human behaviour, hardware failures, …

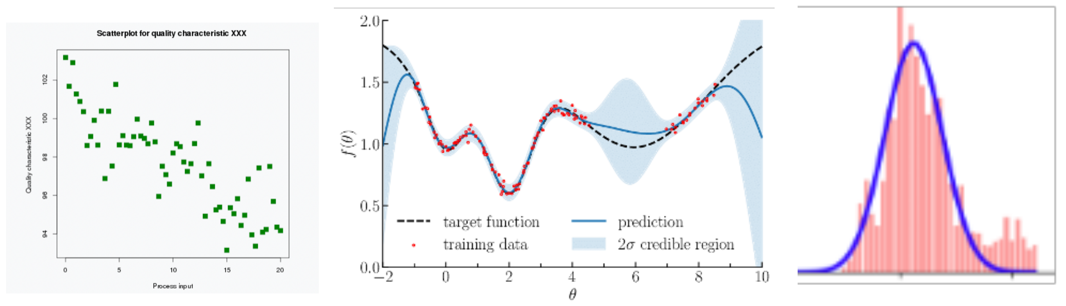
- Trustworthy, safe and robust decision making
  - e.g. for safety-critical applications
  - needs rigorous/systematic quantification of uncertainty



Reasoning about uncertainty

- Markov decision processes (MDPs) and variants
  - standard models for sequential decision making under uncertainty
  - stochastic processes quantify uncertainty
  - but parameters of these often need to be estimated from data





- We will distinguish between:

- Aleatoric uncertainty (randomness intrinsic to environment)
  - e.g., sensor noise, actuator failure, human decisions

(1)

P(H)=P(T)=0.5

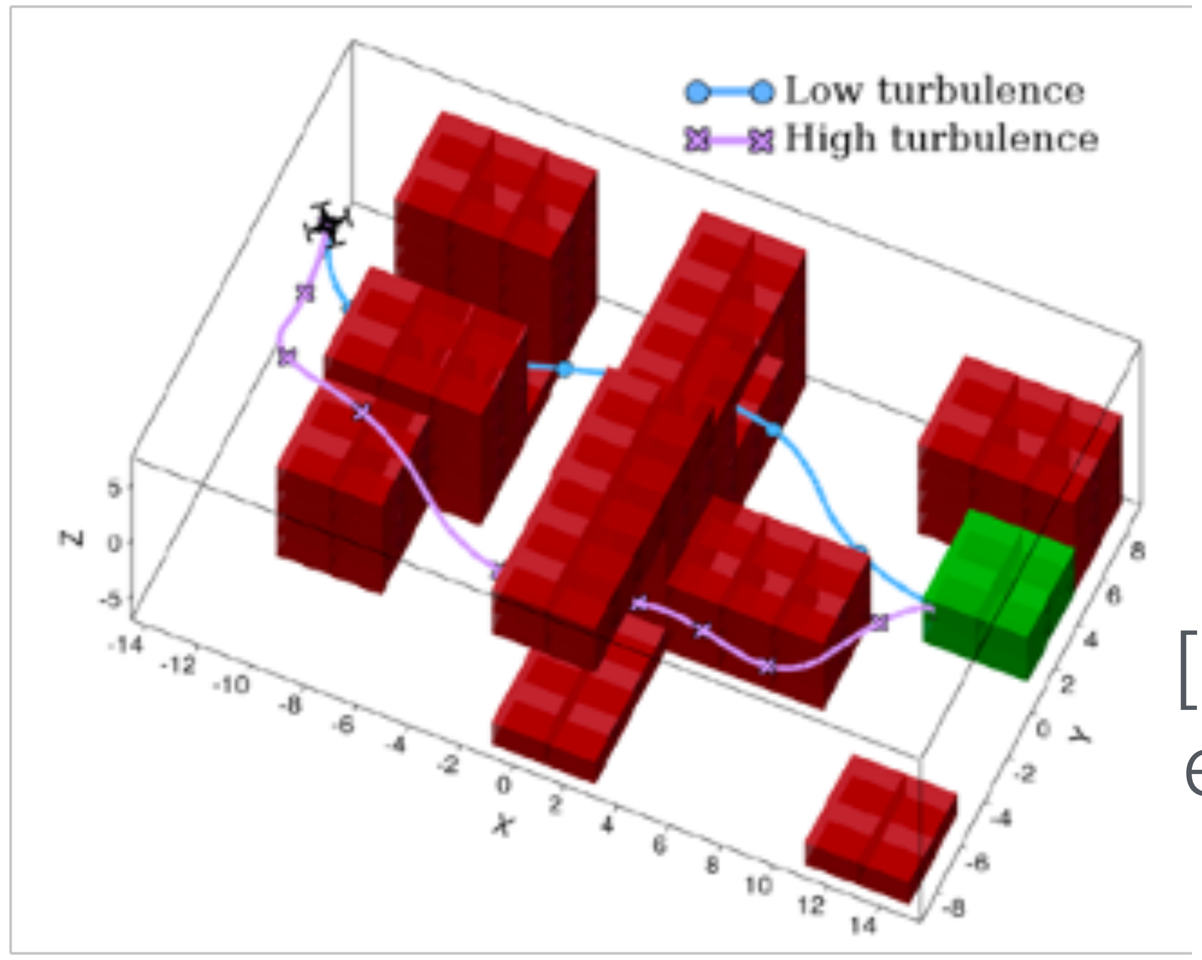
- Epistemic uncertainty (quantifies lack of knowledge)
  - reducible: can reduce by collecting more data/observations
  - e.g., poor model quality due to low number of measurements

(2)

P(H)=1 P(T)=1

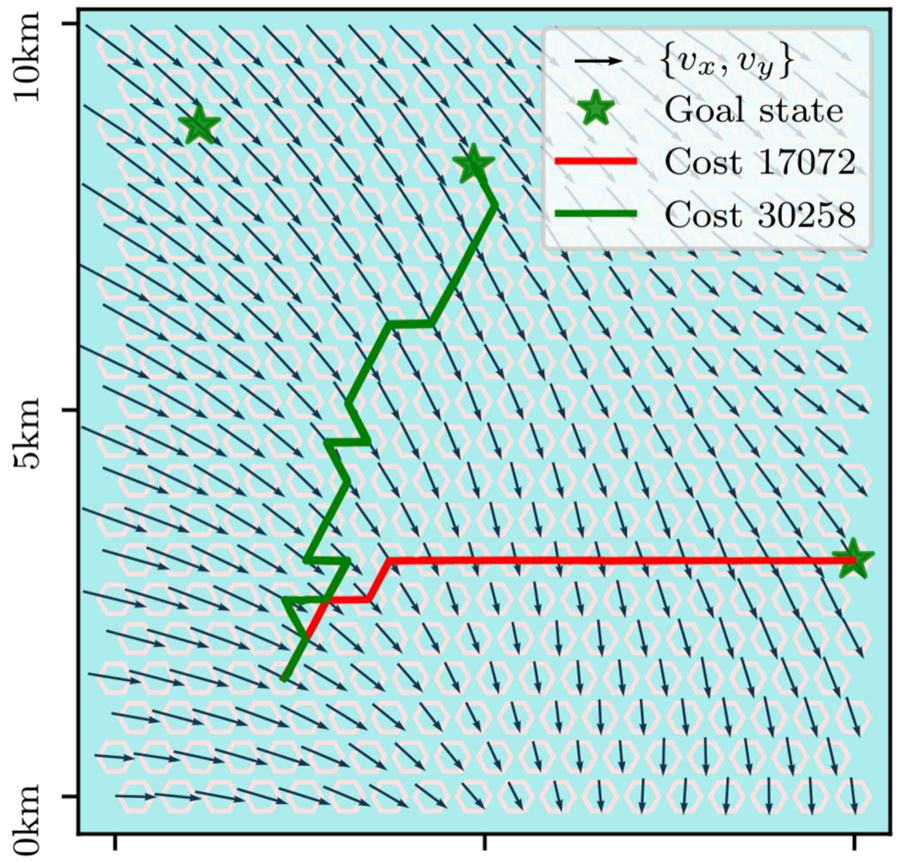
Applications & challenges

- Unmanned aerial vehicle
  - robust control in the presence of turbulence



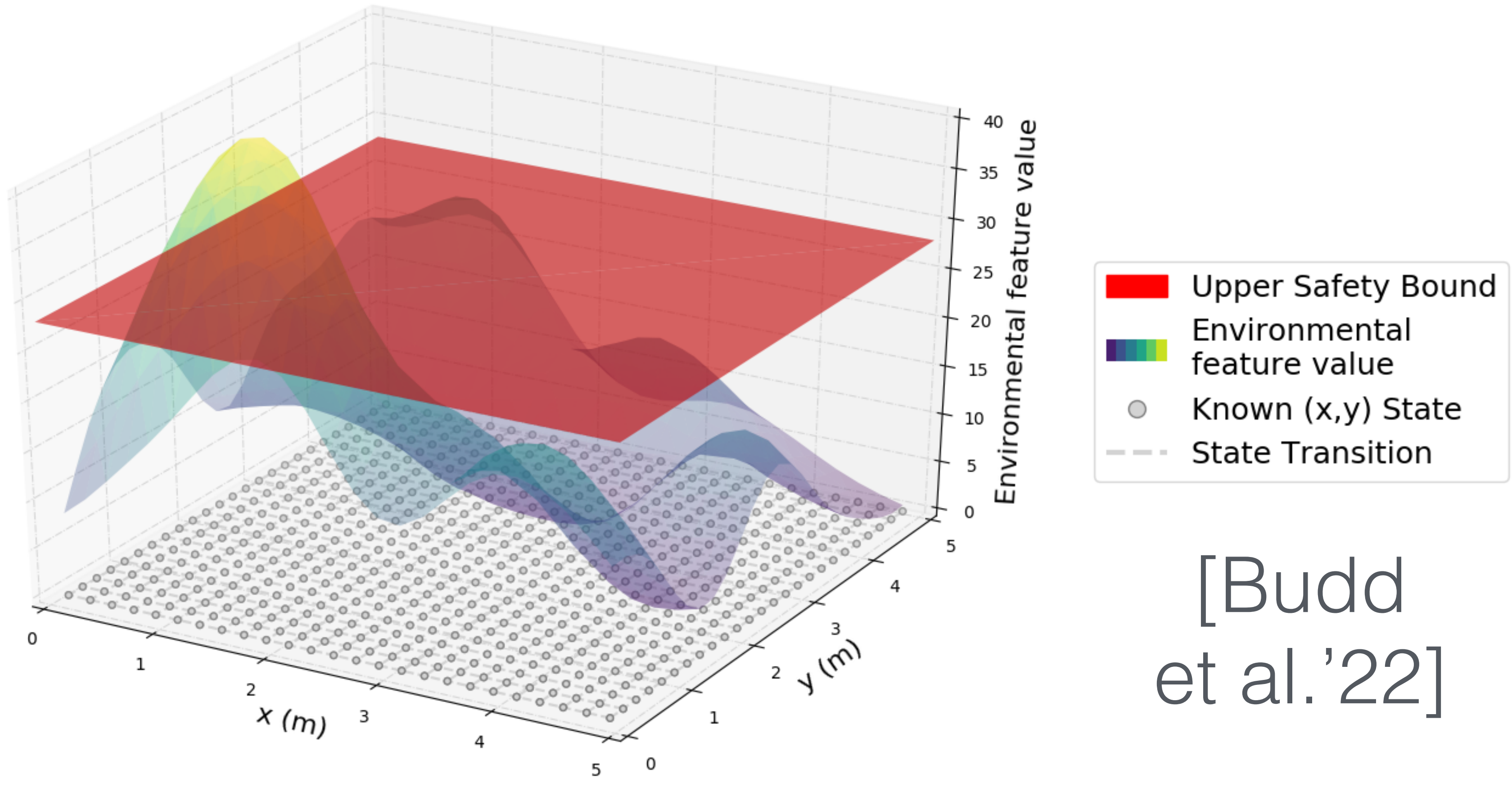
[Badings et al.'23]

- Autonomous underwater vehicle
  - effective navigation against unknown ocean currents



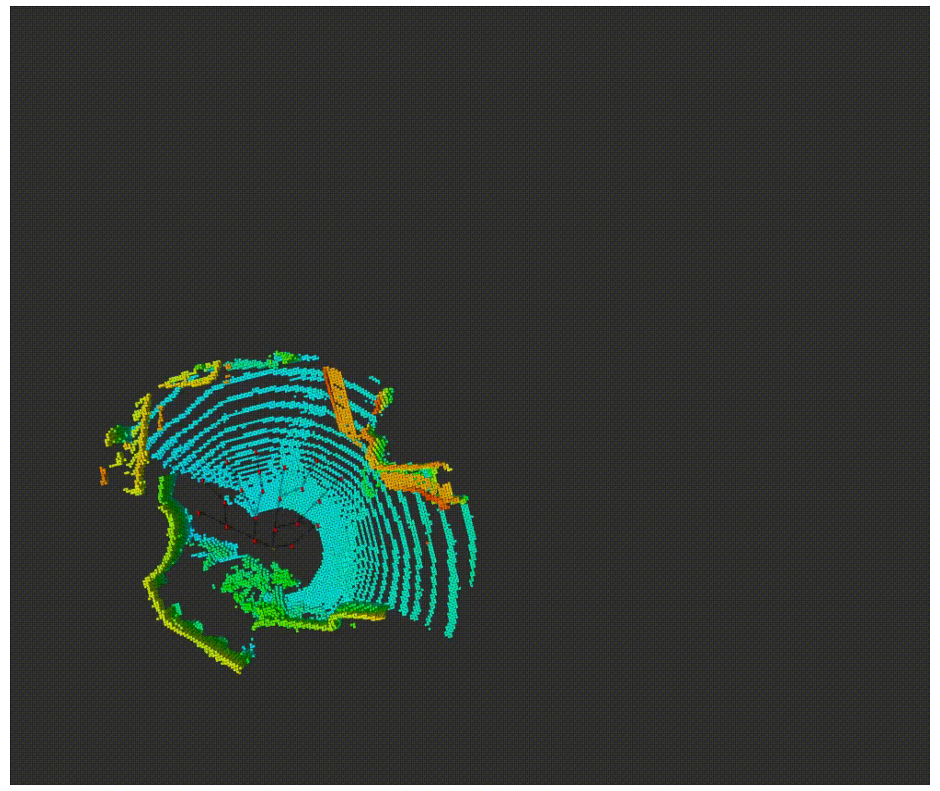
[Budd et al.'22]

- Radiation measuring
  - safe navigation and task completion in unknown environments



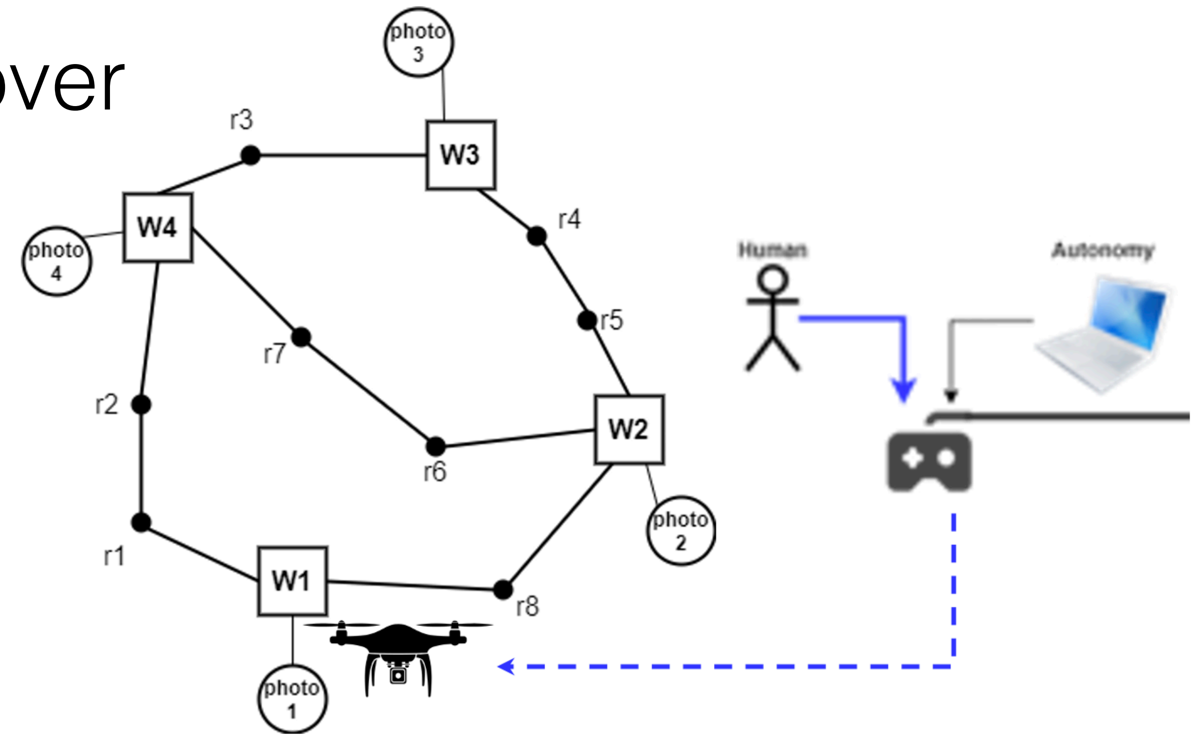
[Budd et al.'22]

- Mine exploration
  - Safe exploration and mapping (avoiding radiation)



- Shared autonomy
  - learning belief over uncertainty on unobservable human state

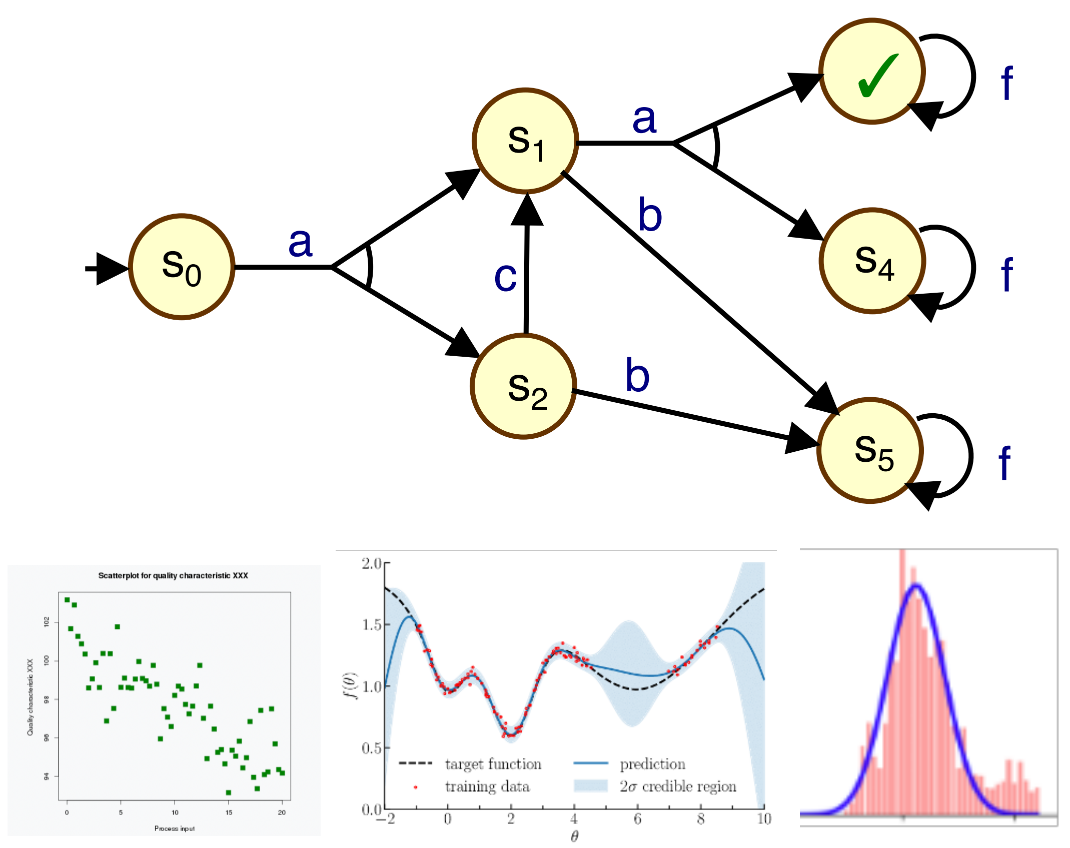
[Costen et al.'22]



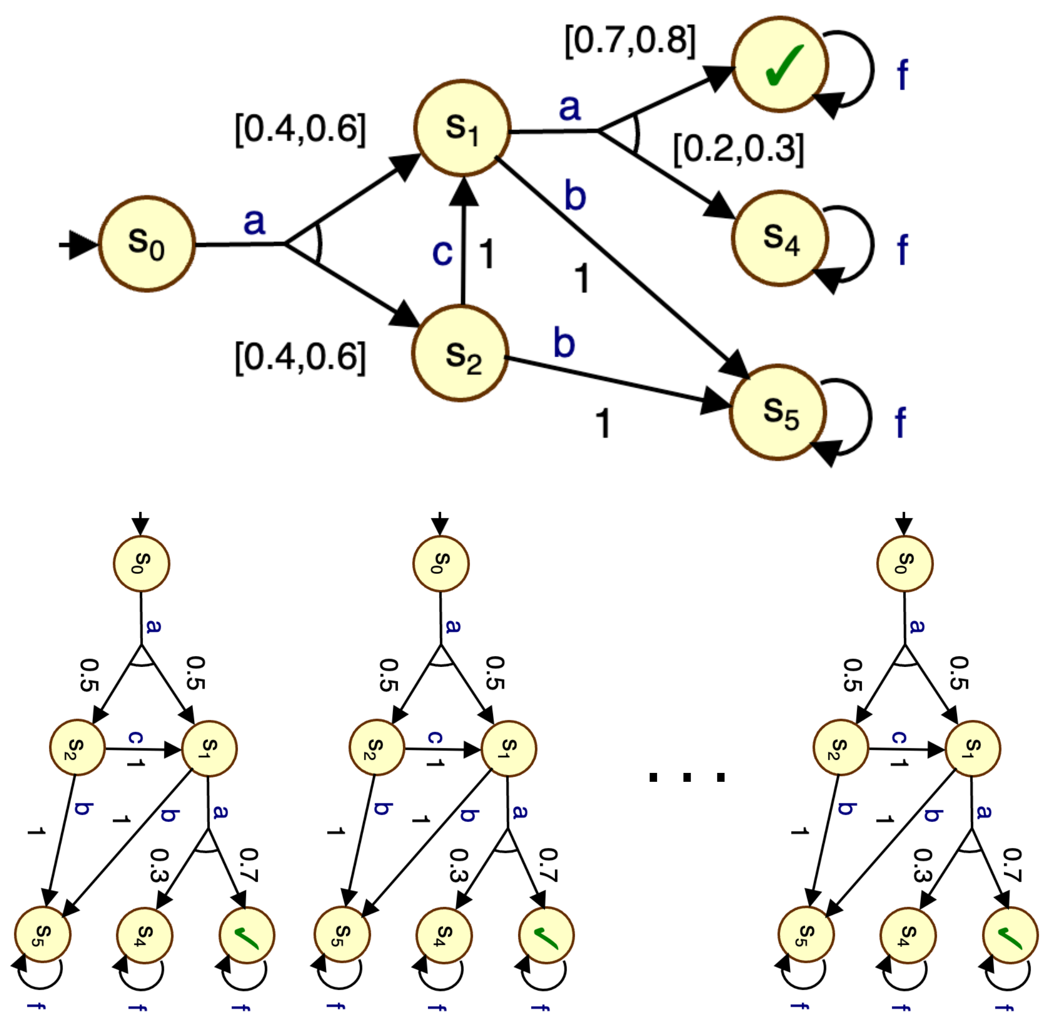
This course

- Model uncertainty in sequential decision making
  - model-based techniques (probabilistic planning, not reinforcement learning)
  - discrete time, discrete space
  - fully observable environments (mostly)
  - rigorous/precise/systematic quantification of uncertainty

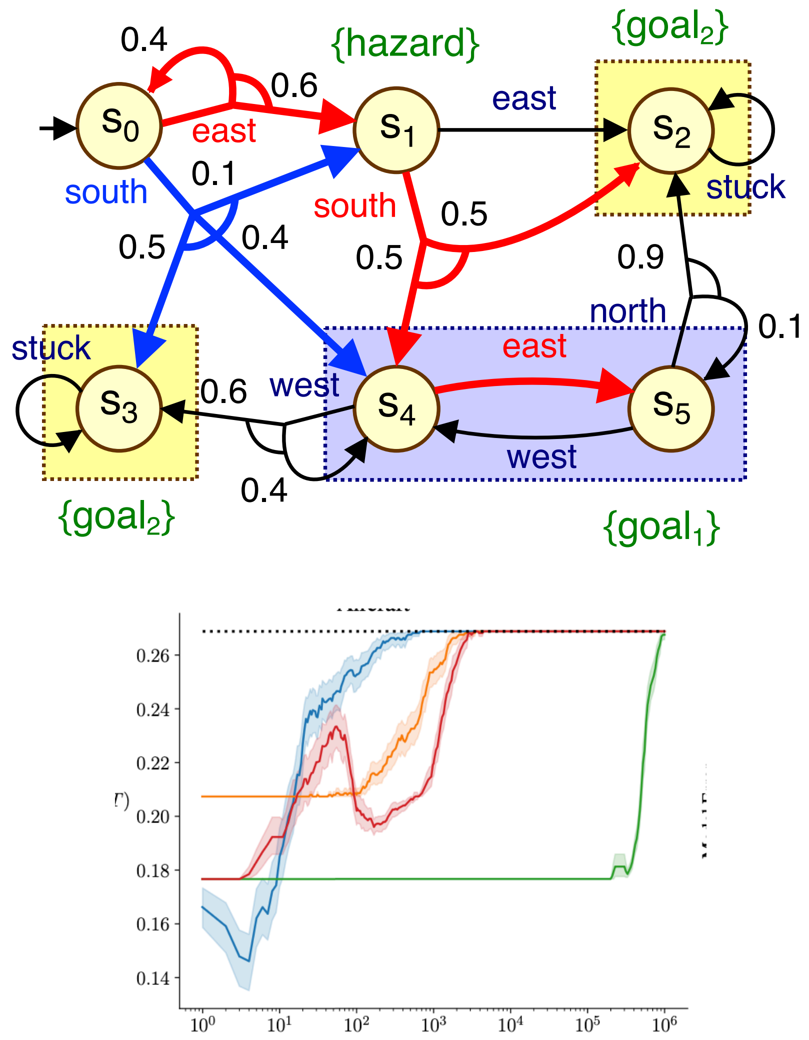
models + data



uncertain MDPs



policies + analysis & guarantees



Course contents

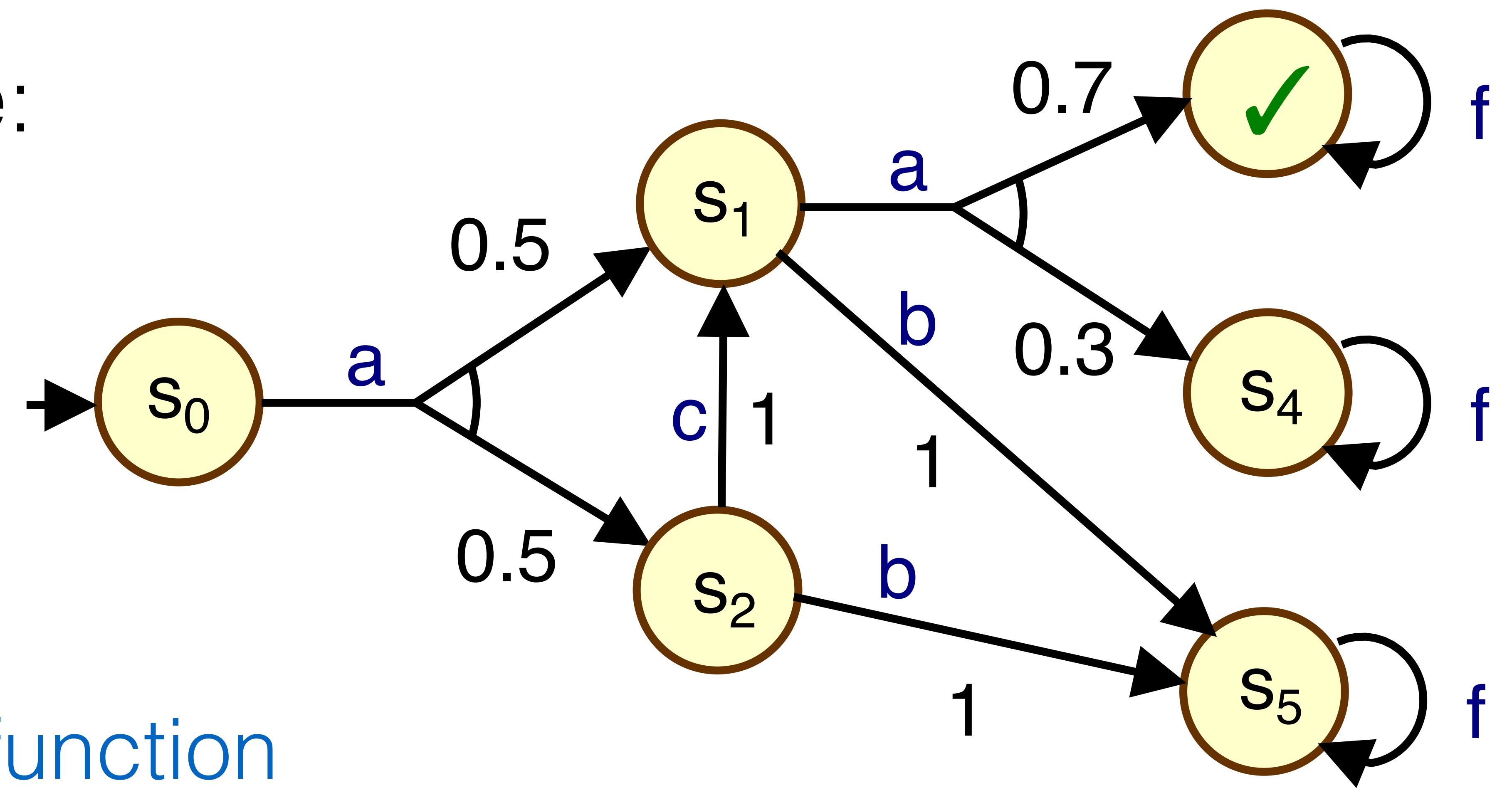
- Markov decision processes (MDPs) and stochastic games
    - MDPs: key concepts and algorithms
    - stochastic games: adding adversarial aspects
- Uncertain MDPs
    - MDPs + epistemic uncertainty, robust control, robust dynamic programming, interval MDPs, uncertainty set representation, challenges, tools
- Sampling-based uncertain MDPs
    - removing the transition independence assumption
- Bayes-adaptive MDPs
    - maintaining a distribution over the possible models

Markov decision processes

Markov decision processes

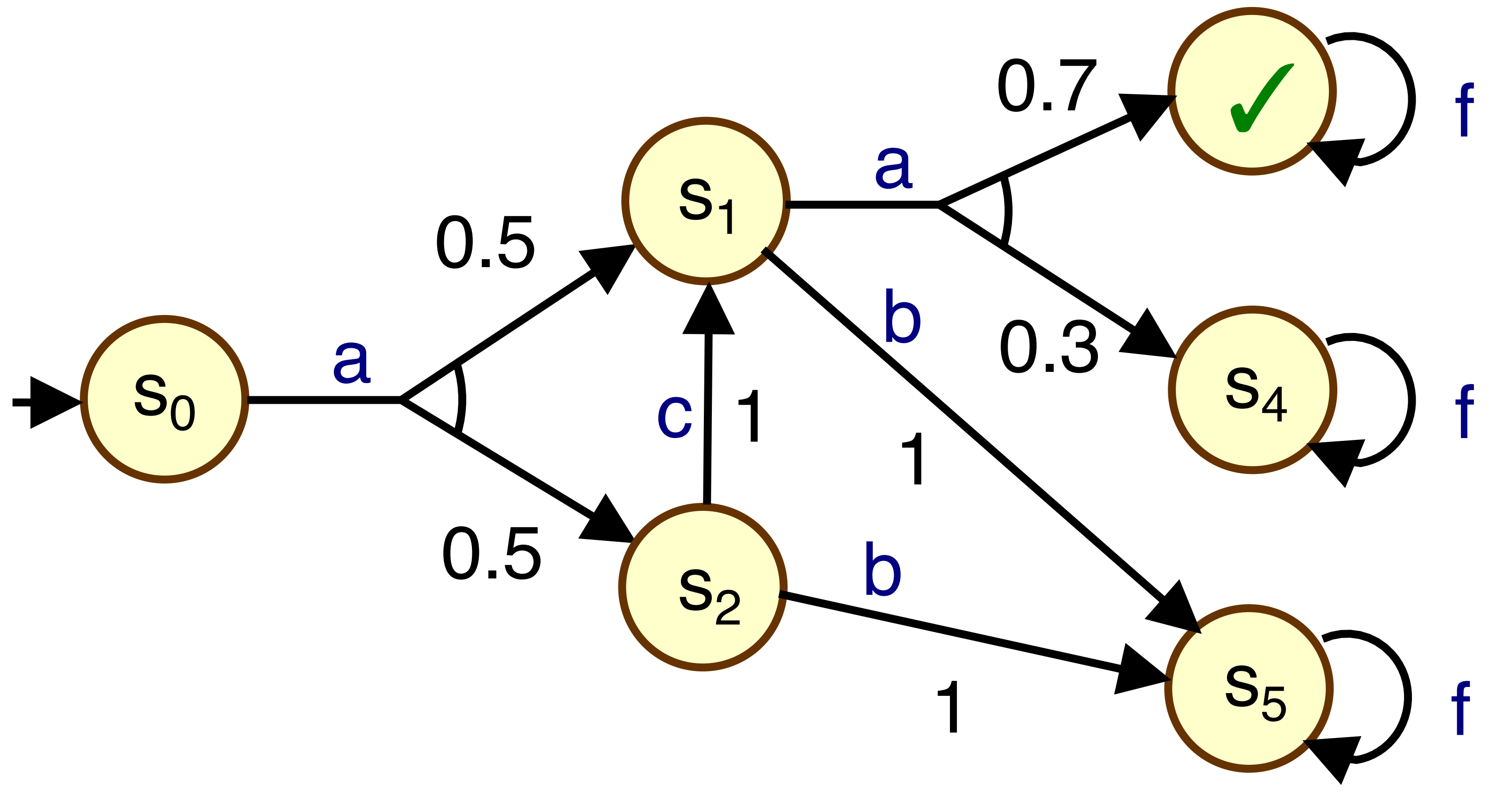
- Markov decision processes (MDPs)
  - standard model for sequential decision making under uncertainty

- An MDP is of the form $\mathcal{M} = (S, s_0, A, P)$ where:
  - S is a (finite) set of states
  - $s_0 \in S$ is an initial state
  - A is a (finite) set of actions
  - $P : S \times A \times S \to [0,1]$ is a transition probability function
    - where $\Sigma_{s' \in S} P(s, a, s') \in \{0,1\}$



Markov decision processes

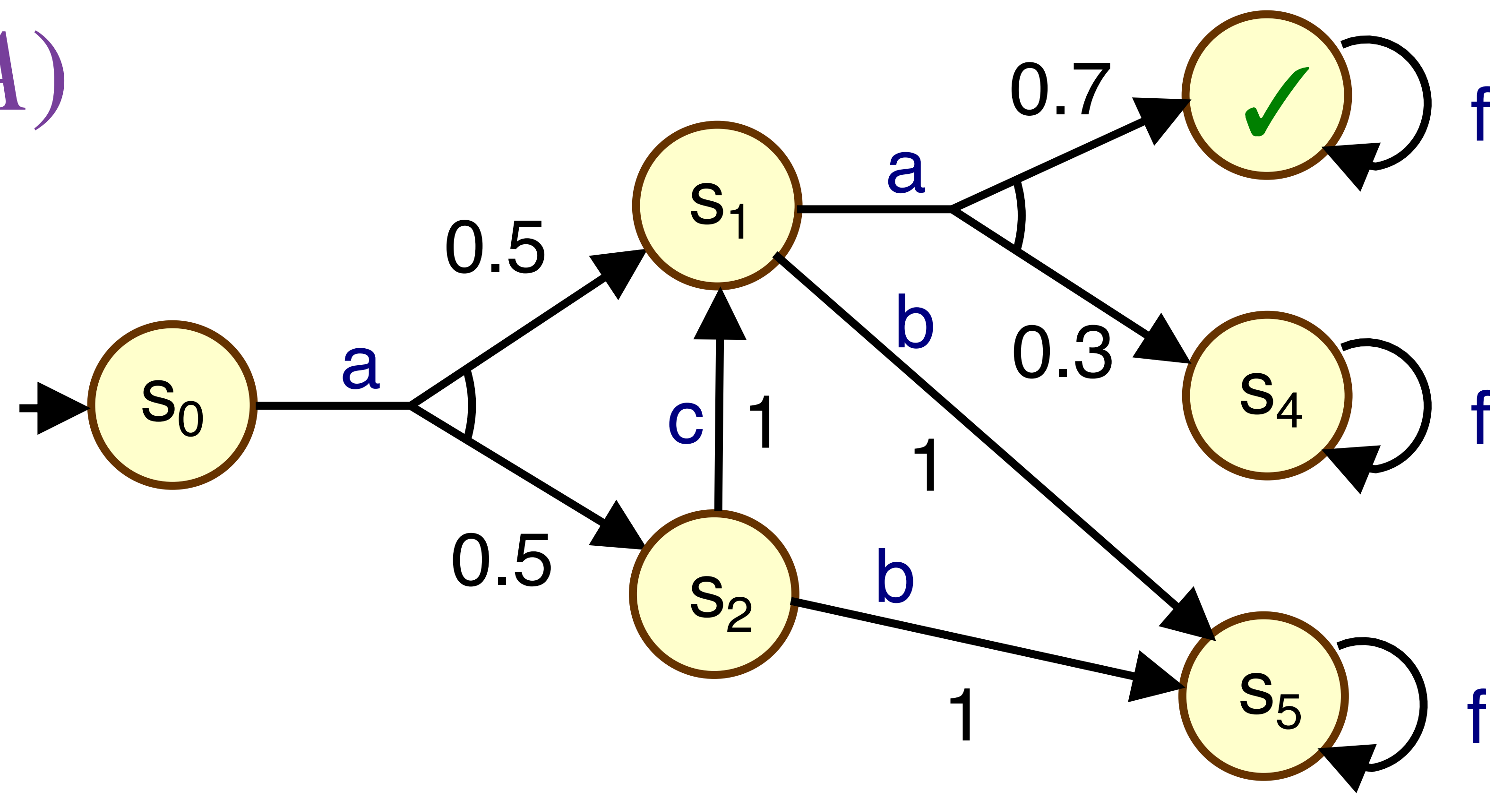
- For an MDP $\mathscr{M} = (S, s_0, A, P)$:
    - the enabled actions $A(s) \subseteq A$ in each state s
        - are $A(s) = \{a \in A : P(s, a, s') > 0$ for some $s'\}$
    - a path is a sequence $\omega = s_0 a_0 s_1 a_1, \ldots$
        - such that $s_i \in S$, $a_i \in A(s_i)$ and $P(s_i, a_i, s_{i+1}) > 0$ for all i



- We also use:
    - $P^a : S \times S \to [0,1]$ is the transition probability matrix for each $a \in A$
    - $P^a_s \in Dist(S)$ is the successor distribution for each state s and action $a \in A(s)$
    - (where $Dist(S)$ is the set of discrete probability distributions over set S)

Policies for MDPs

- Policies (or strategies) π resolves the choice of action in each state
  - based on the execution of the MDP so far
  - formally: a policy is a mapping $\pi : (S \times A)^* \times S \rightarrow Dist(A)$
    - such that $\pi(s_0 a_0 \ldots s_n)(a_n) > 0$ implies $a_n \in A(s_n)$
  - $\pi(s_0 a_0 \ldots s_n)(a_n)$ is the probability of picking a_n after observing MDP history $s_0 a_0 \ldots s_n$



- $\Pi_{\mathscr{M}}$ (or just Π) is the set of all (deterministic) policies for MDP $\mathscr{M}$

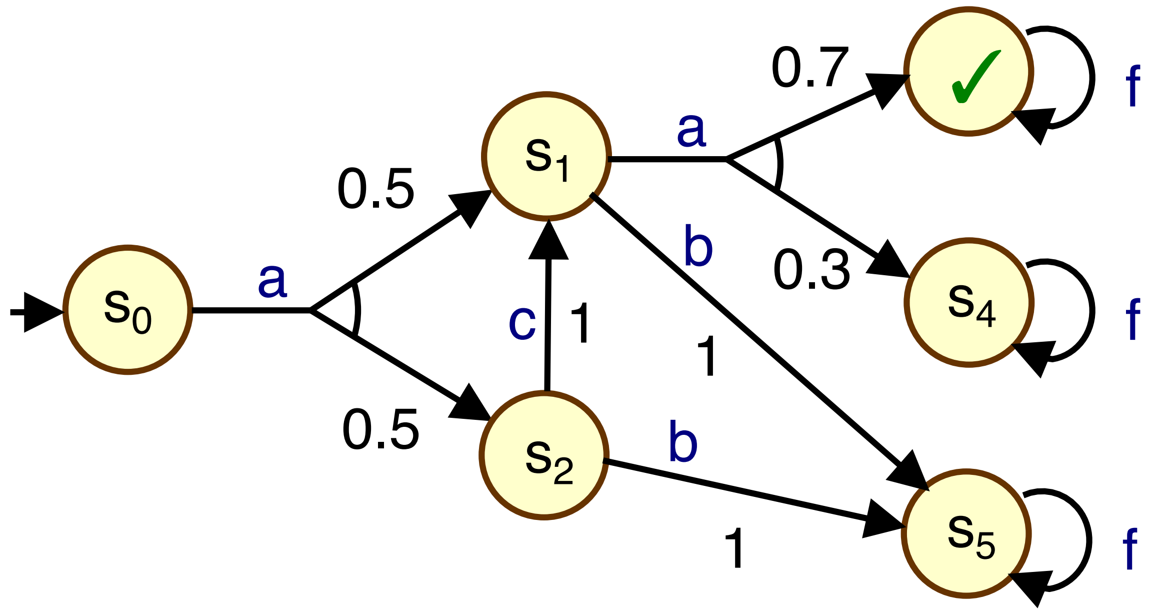
- Policies can be classified by (i) use of randomisation; (ii) use of memory
  - which matter for optimality, computation, practicality, …

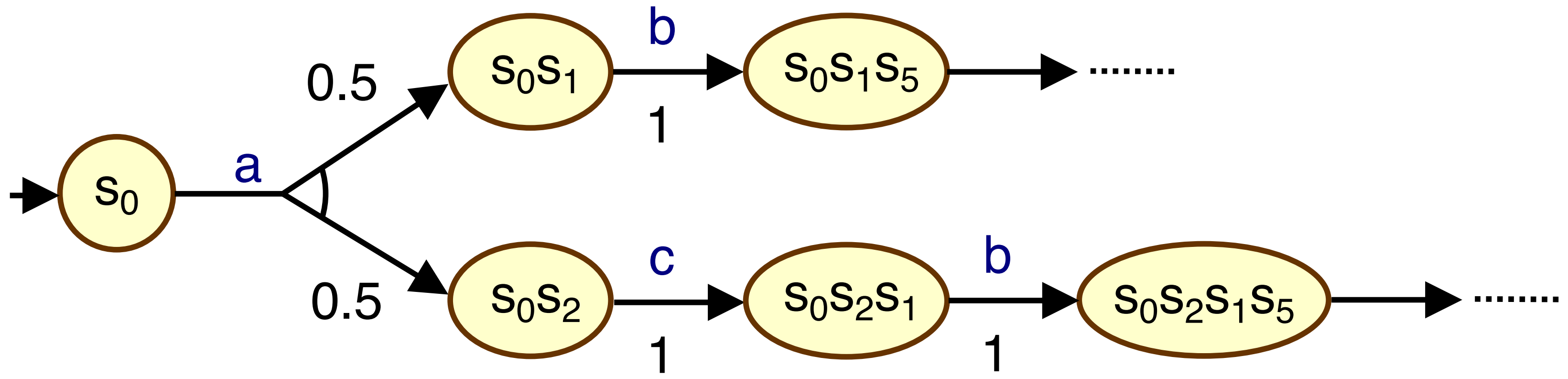
Classes of policies for MDPs

- Randomisation
  - π is deterministic (or pure) if it always picks a single action with probability 1
  - and randomised (or probabilistic) otherwise
  - for now, we'll mostly assume deterministic policies and assume $\pi : (S \times A)^* \times S \to A$
- Memory
  - π is memoryless (or stationary, or Markovian) if $\pi(s_0, \ldots, s_n) = \pi(s'_0, \ldots, s'_n)$ when $s_n = s'_n$
    - in which case we write it in the form $\pi : S \to A$
    - $\Pi_m \subseteq \Pi$ is the set of all memoryless policies
  - otherwise π is history dependent
  - π is finite-memory if it suffices to distinguish a finite number of "modes" based on the history
  - sometimes write a (time-dependent) policy as tuple $\pi = (\pi_0, \pi_1, \ldots)$ where $\pi_i : S \to A$

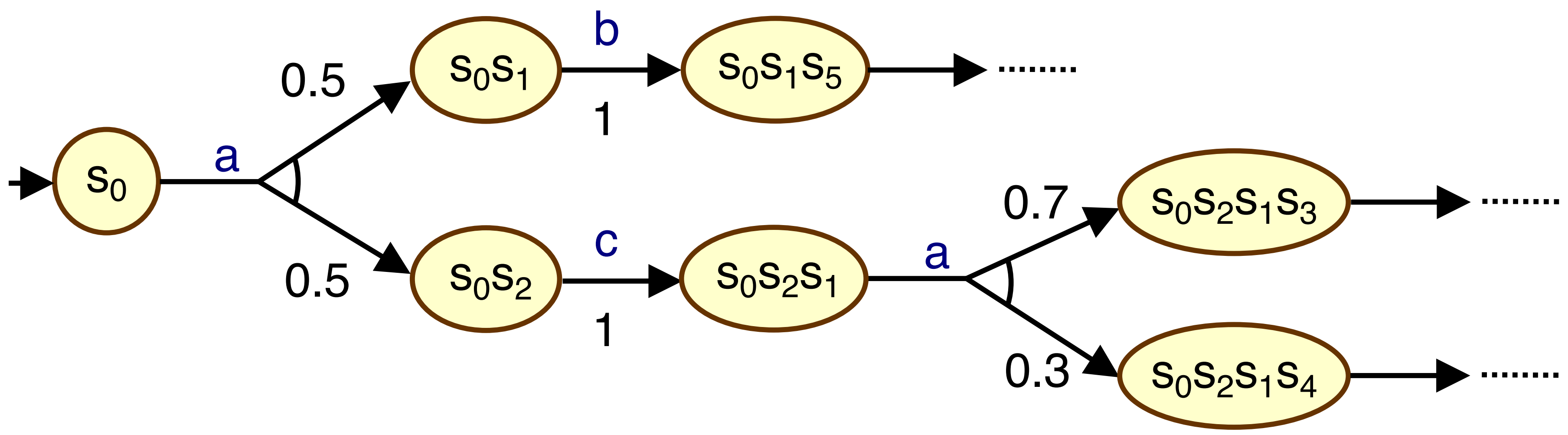
MDPs and policies

- A policy for an MDP yields an induced Markov chain
  - and set of (infinite) paths

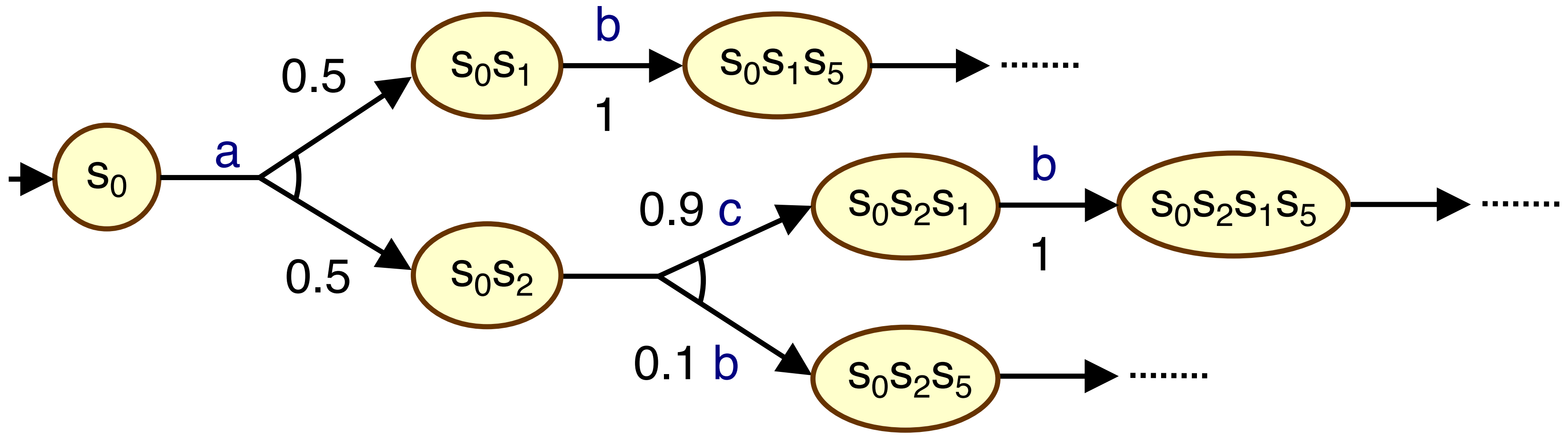




(memoryless, deterministic)



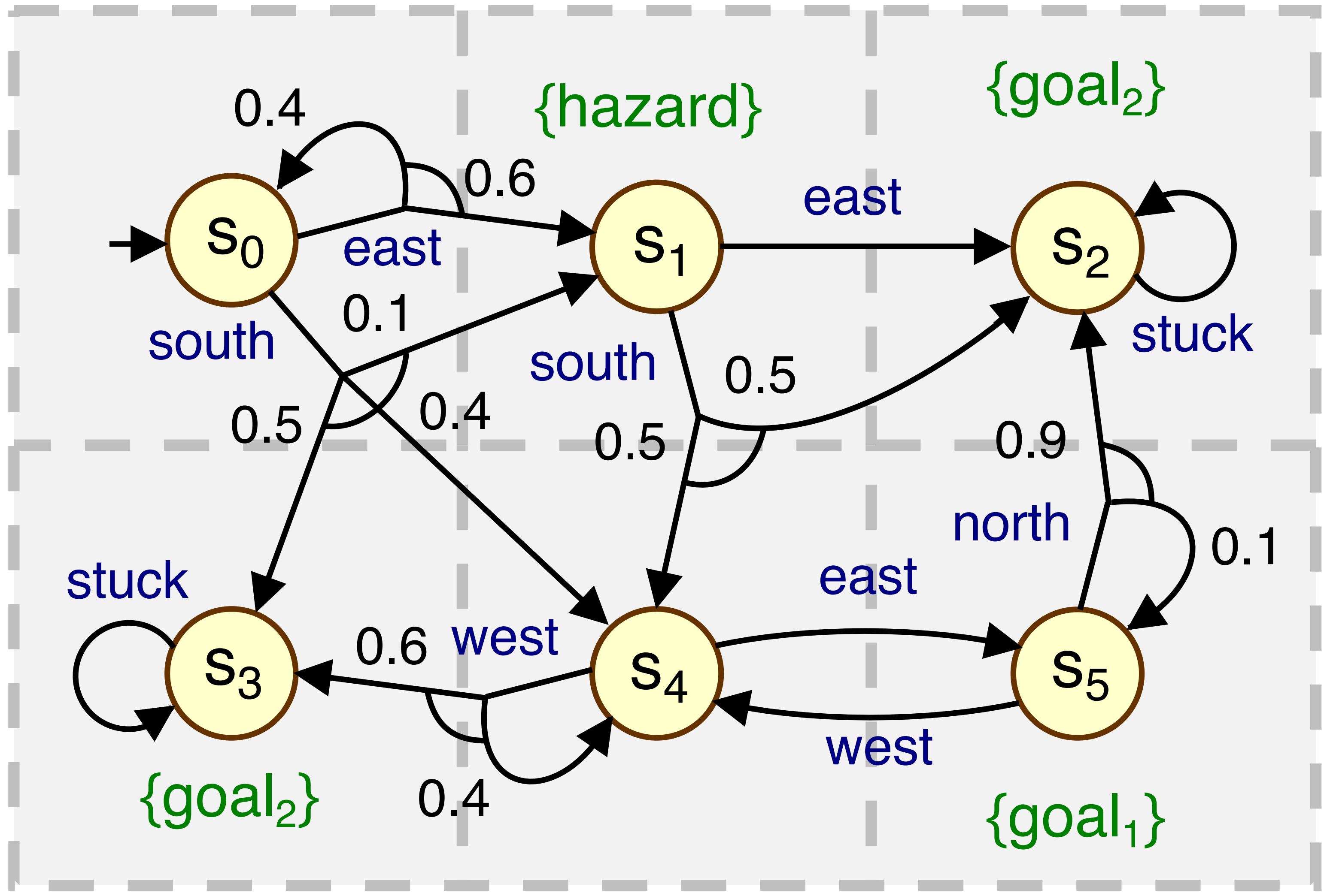
(finite-memory, deterministic)



(memoryless, randomised)

Running example (and objectives)

- Example MDP: robot moving through terrain divided in to 3 x 2 grid



- Objectives (or properties) define an optimisation problem for an MDP
  - MaxProb: maximise the probability of reaching $goal \subseteq S$
  - SSP (stochastic shortest path): minimise the cost of reaching $goal \subseteq S$

} we'll focus mainly on these two

Defining objectives for MDPs

- Execution of an MDP under a policy
  - for a policy $\pi \in \Pi$ on MDP $\mathcal{M}$…
  - Pr_s^π is a probability measure over all (infinite) paths from state s of $\mathcal{M}$
  - $\mathbb{E}_s^\pi(X)$ is the expected value of X (with respect to Pr_s^π)
    - where $X : (S \times A)^\omega \rightarrow \mathbb{R}_{\geq 0}$ is a random variable over (infinite) paths

- Value function: $V^\pi : S \rightarrow \mathbb{R}$
  - gives the value of an objective under π starting from each state of the MDP
  - define optimal value, e.g.: $V^*(s) = \max_{\pi \in \Pi} V^\pi(s)$
  - and optimal policy, e.g.: $\pi^* = \text{argmax}_{\pi \in \Pi} V^\pi(s_0)$

MaxProb & SSP (stochastic shortest path)

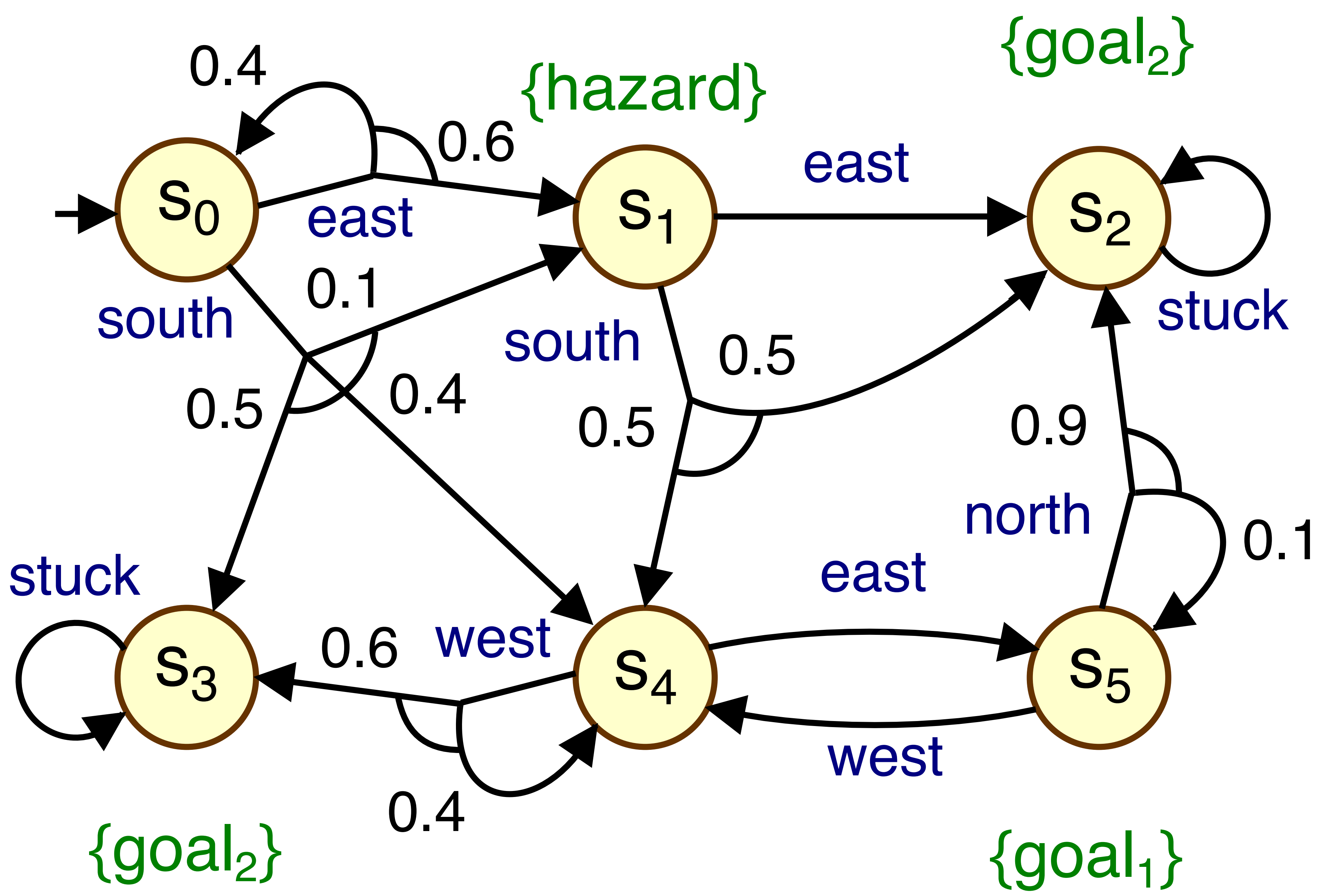
- MaxProb: Maximise the probability of reaching a target state set $goal \subseteq S$
  - maximise $V^\pi(s) = Pr_s^\pi(\{s_0 a_0 s_1 a_1 s_2 \ldots : s_i \in goal$ for some $i\})$

- SSP: Minimise the expected cost of reaching a target state set $goal \subseteq S$
  - for a cost function $C : S \times A \to \mathbb{R}_{\geq 0}$
  - minimise $V^\pi(s) = \mathbb{E}_s^\pi(X^C)$ where $X^C(s_0 a_0 s_1 a_1 \ldots) = \Sigma_{i=0}^\infty C(s_i, a_i)$

- Assumptions for SSP
  - $goal$ states are absorbing and zero-cost
  - there is a proper policy (i.e., which reaches $goal$ with probability 1 from all states)
  - every improper policy incurs an infinite cost from every state from which it does not reach $goal$ with probability 1

Running example: MaxProb

- What is the optimal policy for objective MaxProb($\color{green}{goal_1}$)?



Other objectives

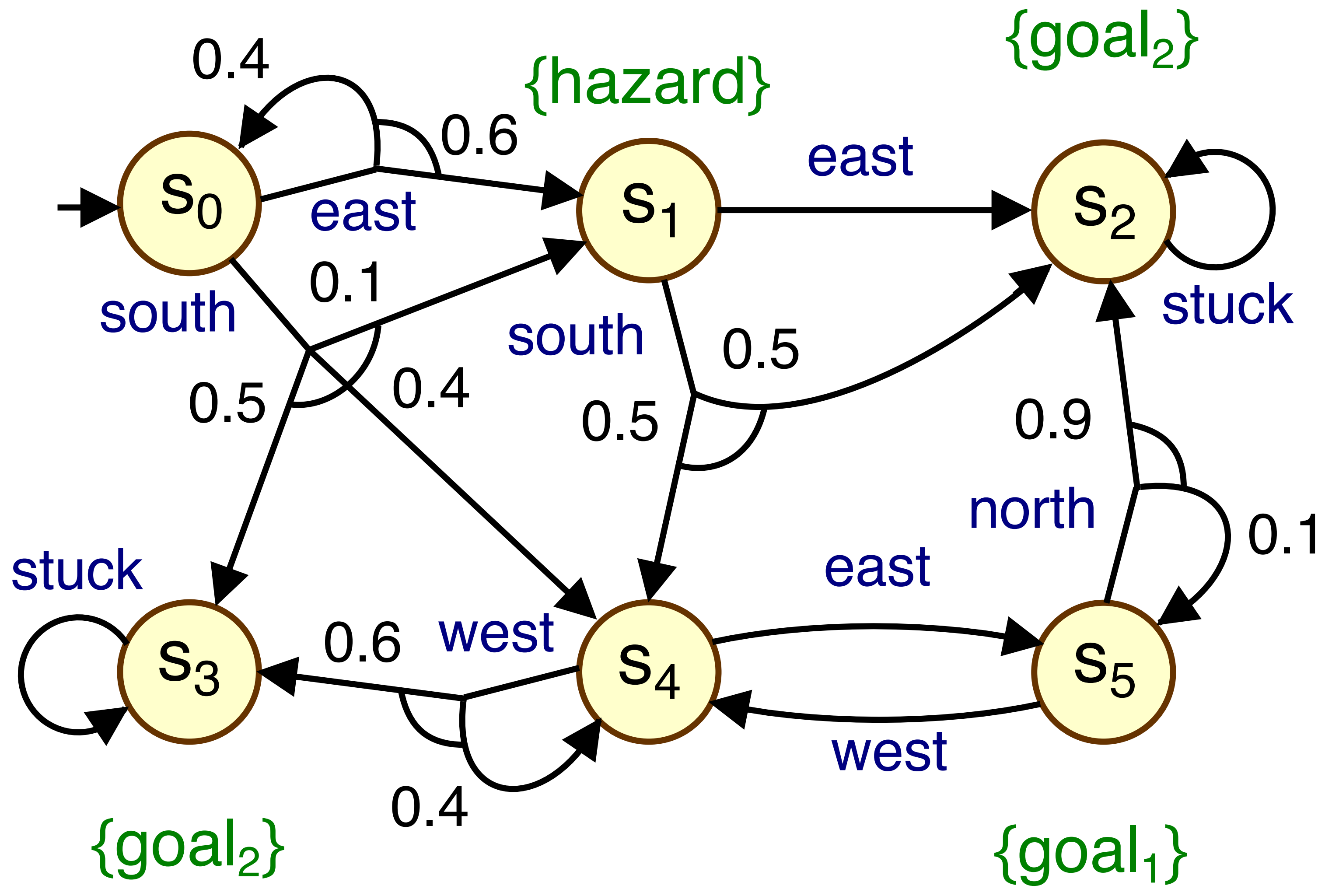
- Some other common objectives for MDPs:

- Finite-horizon variants, e.g., of MaxProb:
  - MaxProb$^{\leq k}$: Maximise the probability of reaching $goal \subseteq S$ within time horizon k
  - maximise $V^\pi(s) = Pr_s^\pi(\{s_0 a_0 s_1 a_1 s_2 \ldots : s_i \in goal$ for some $i \leq k\})$

- Discounting infinite-horizon objectives
  - DiscSum: Maximise the expected discounted total reward sum
  - for a reward function $R : S \times A \to \mathbb{R}$ and discount factor $\gamma \in (0,1)$
  - maximise $V^\pi(s) = \mathbb{E}_s^\pi(X^R)$ where $X^R(s_0 a_0 s_1 a_1 \ldots) = \Sigma_{i=0}^{\infty} \gamma^i R(s_i, a_i)$

Temporal logic objectives

- Specification languages from formal verification
  - probabilistic extensions of temporal logics, e.g., PCTL, PLTL



- Examples
  - $P_{max=?}$ [F goal$_1$] - "probabilistic reachability"
  - $P_{max=?}$ [$F^{\leq 10}$ goal$_1$] - "probabilistic bounded reachability"
  - $P_{max=?}$ [G ¬hazard] - "probabilistic safety/invariance"
  - $P_{max=?}$ [¬hazard U goal$_1$] - "probabilistic reach-avoid"
  - $P_{max=?}$ [(G¬hazard) ∧ (GF goal$_1$)] - "maximise probability of avoiding hazard and also visiting goal 1 infinitely often"
  - $P_{max=?}$ [¬zone$_3$ U (zone$_1$ ∧ (F zone$_4$))] - "maximise probability of patrolling zone 1 (whilst avoiding zone 3) then zone 4"
  - $R_{time,min=?}$ [¬zone$_3$ U (zone$_1$ ∧ (F zone$_4$))] - "minimise the expected time to patrol zone 1 (whilst avoiding zone 3) then zone 4"

Solving MDPs

- We will mainly focus on MaxProb (techniques are very similar for SSP)
- Key result: memoryless (deterministic) policies suffice

$$\max_{\pi \in \Pi} V^{\pi}(s) = \max_{\pi \in \Pi_m} V^{\pi}(s)$$

- The optimal value function satisfies the Bellman equation:

$$V^*(s) = \begin{cases} 1 & \text{if } s \in goal \\ \max_{a \in A(s)} \sum_{s' \in S} P_s^a(s') \cdot V^*(s') & \text{otherwise} \end{cases}$$

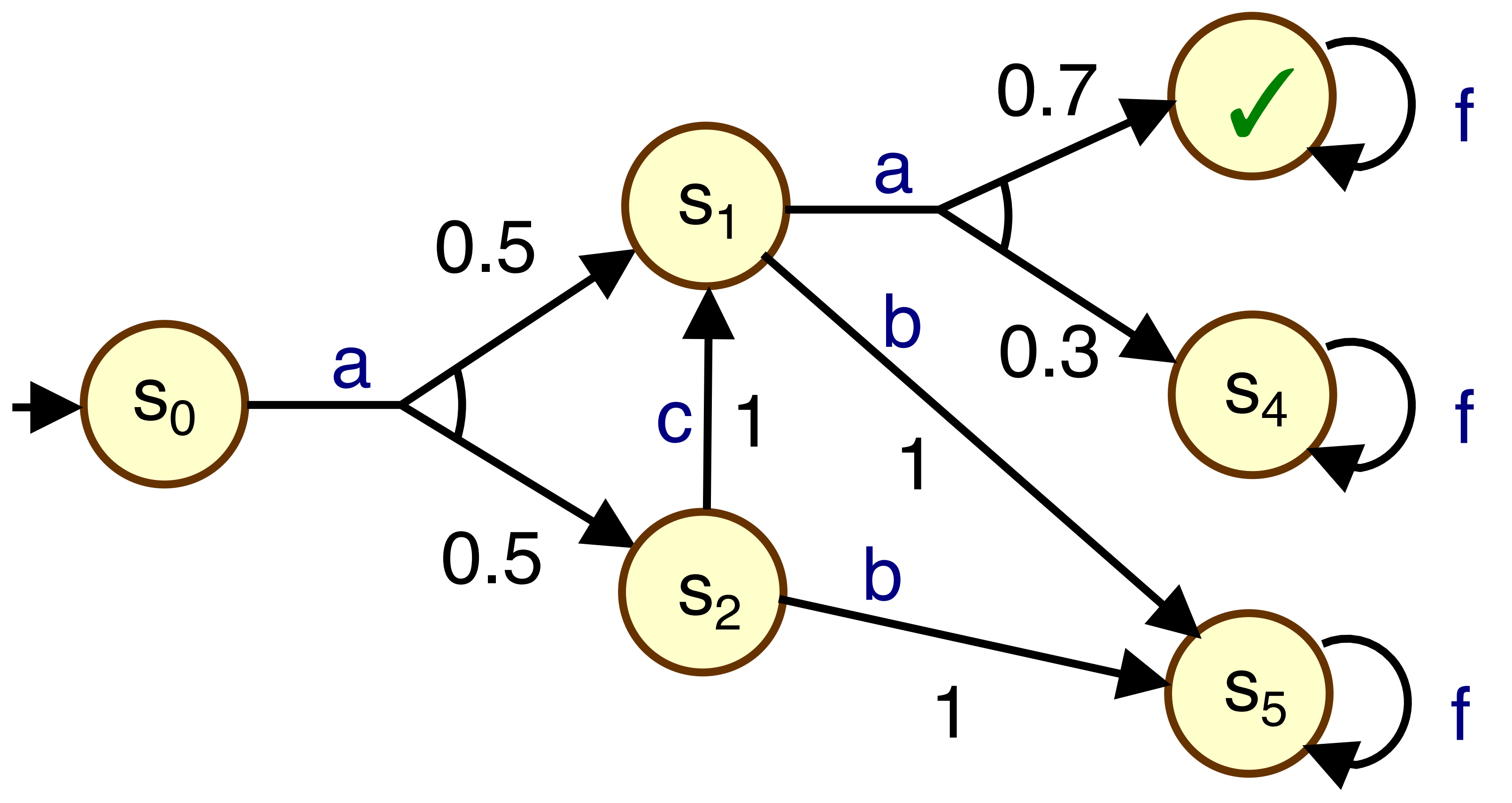
- Solution methods
  - value iteration (dynamic programming)
  - linear programming
  - and many more (e.g., policy iteration, Monte Carlo tree search, BRTDP, …)

MaxProb via value iteration

- Optimal values can be obtained using dynamic programming
  - from the limit of the vector sequence defined below
  - $V^*(s) = \lim_{k\to\infty} x_s^k$ where:

$$x_s^k = \begin{cases} 1 & \text{if } s \in goal \\ 0 & \text{if } s \notin goal \text{ and } k = 0 \\ \max_{a\in A(s)} \sum_{s'\in S} P_s^a(s') \cdot x_{s'}^{k-1} & \text{otherwise} \end{cases}$$

Bellman backup operator



- Known as value iteration (VI)
  - the Bellman operator is (i) monotonic (ii) a contraction in the L_∞ norm
  - optimal values are the least fixed point of the Bellman operator

MaxProb via value iteration

- Optimise via graph-based pre-computation
  - potentially improves accuracy / convergence, resolves uniqueness
  - compute state sets:
    - S^0 = (all) states for which all policies reach $goal$ *with probability 0 (i.e., max = 0)*
    - $S^1 \supseteq goal$ = (some) states for which a policy reaches $goal$ *with probability 1 (i.e., max = 1)*
    - $S^? = S \backslash (S^0 \cup S^1)$

- Then value iteration becomes:

$$x_s^k = \begin{cases} 1 & \text{if } s \in S^1 \\ 0 & \text{if } s \in S^0 \\ 0 & \text{if } s \in S^? \text{ and } k = 0 \\ \max_{a \in A(s)} \sum_{s' \in S} P_s^a(s') \cdot x_{s'}^{k-1} & \text{otherwise} \end{cases}$$

Implementation details:

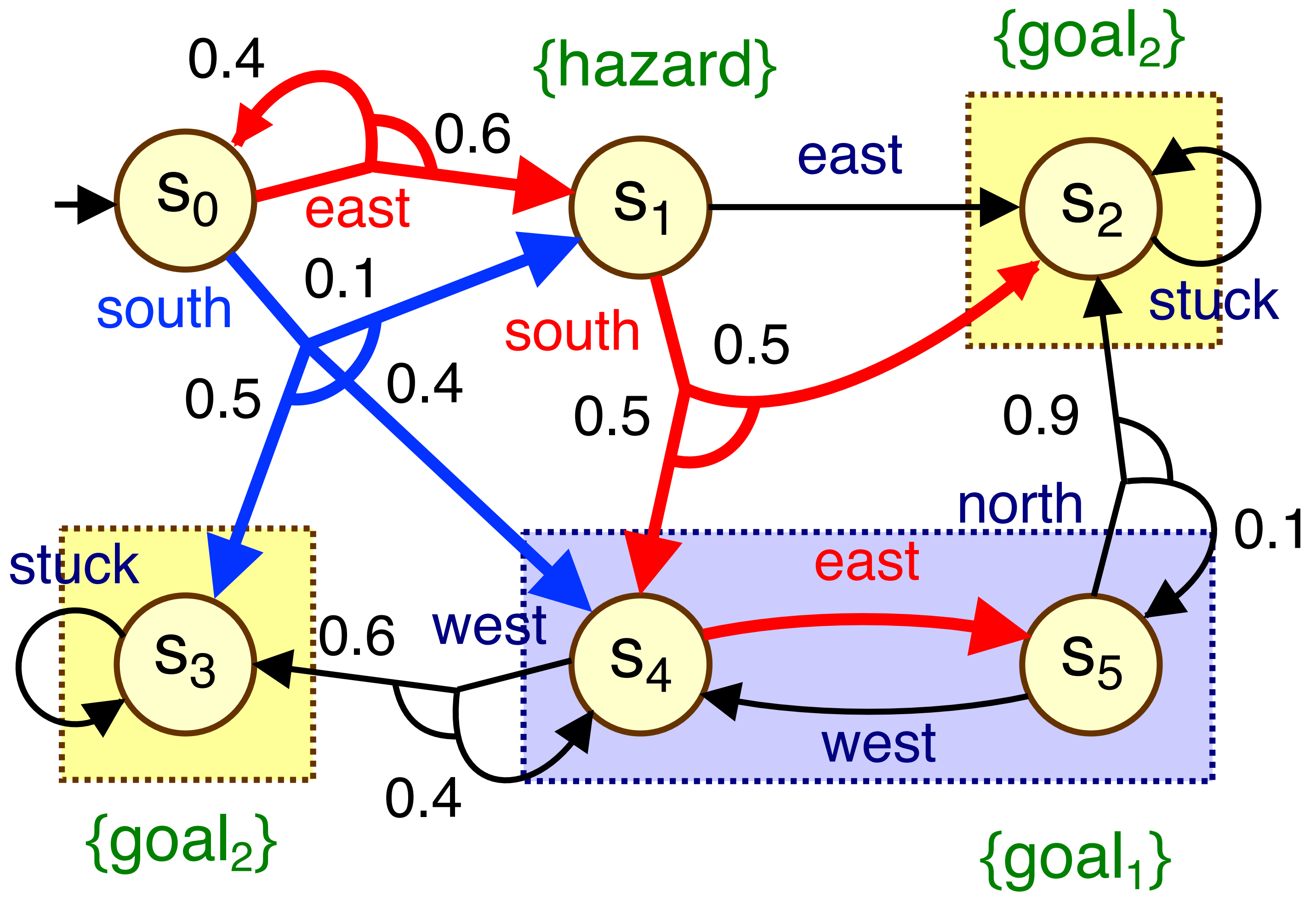
- Extract optimal policy after/during:

$$\pi^*(s) = \operatorname{argmax}_{a \in A(s)} \sum_{s' \in S} P_s^a(s') \cdot x_{s'}^{k-1}$$

- Terminate when $\| x^{k+1} - x^k \| < \varepsilon$
- Choose order to update states s

Running example: Value iteration

- Example: MaxProb($goal_1$)



k	x_0	x_1
0	0	0
1	0.4	0.5
2	0.46	0.5
3	0.484	0.5
4	0.4936	0.5
5	0.49744	0.5
6	0.498976	0.5
7	0.4995904	0.5
8	0.49983616	0.5
9	0.499934464	0.5
10	0.4999737856	0.5

- Fix $x_4=x_5=1$ and $x_2=x_3=0$, just solve for x_0, x_1
- Iteration k=0: $x_0=x_1=0$
- Iteration k=1: $x_0 := \max(0.4\cdot 0 + 0.6\cdot 0,\ 0.1\cdot 0 + 0.5\cdot 0 + 0.4\cdot 1)$
  $= \max(0, 0.4)$
  $= 0.4$

  $x_1 := \max(1\cdot 0,\ 0.5\cdot 0 + 0.5\cdot 1)$
  $= \max(0, 0.5)$
  $= 0.5$
- Iteration k=2: $x_0 := \max(0.4\cdot 0.4 + 0.6\cdot 0.5,\ 0.1\cdot 0.5 + 0.5\cdot 0 + 0.4\cdot 1)$
  $= \max(0.46, 0.45)$
  $= 0.46$

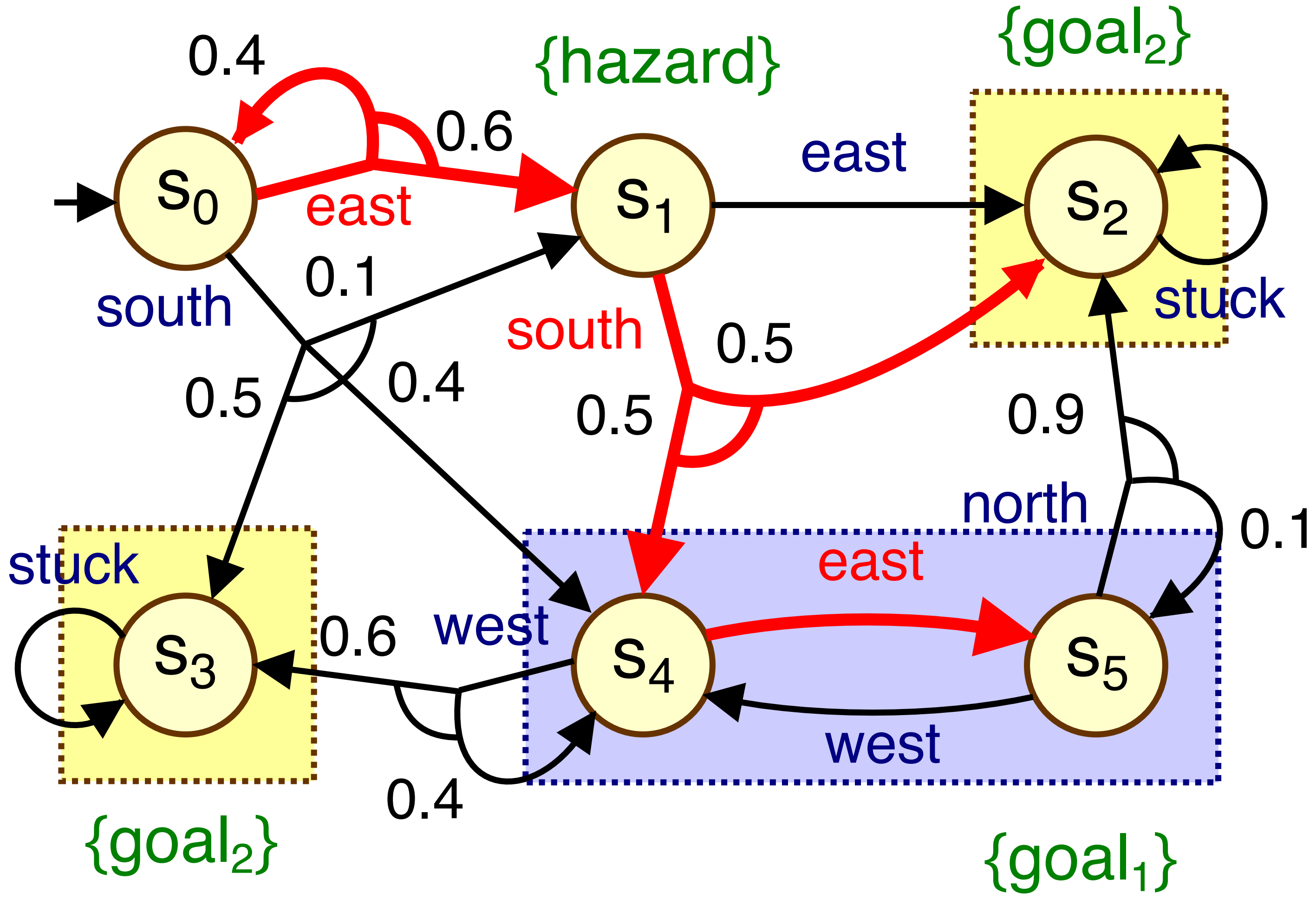
  $x_1 := 0.5$ (as before)
- Finally: $x_0=0.5$, $x_1=0.5$

MaxProb via linear programming

- Optimal values can be computed using linear programming (LP):
  - $V^*(s)$ equals the solution x_s to:

minimise $\Sigma_{s \in S} x_s$ subject to the constraints:

$x_s = 1$	for $s \in S^1$
$x_s = 0$	for $s \in S^0$
$x_s \geq \Sigma_{s' \in S} P_s^a(s') \cdot x_{s'}$	for $s \in S^?$, $a \in A(s)$

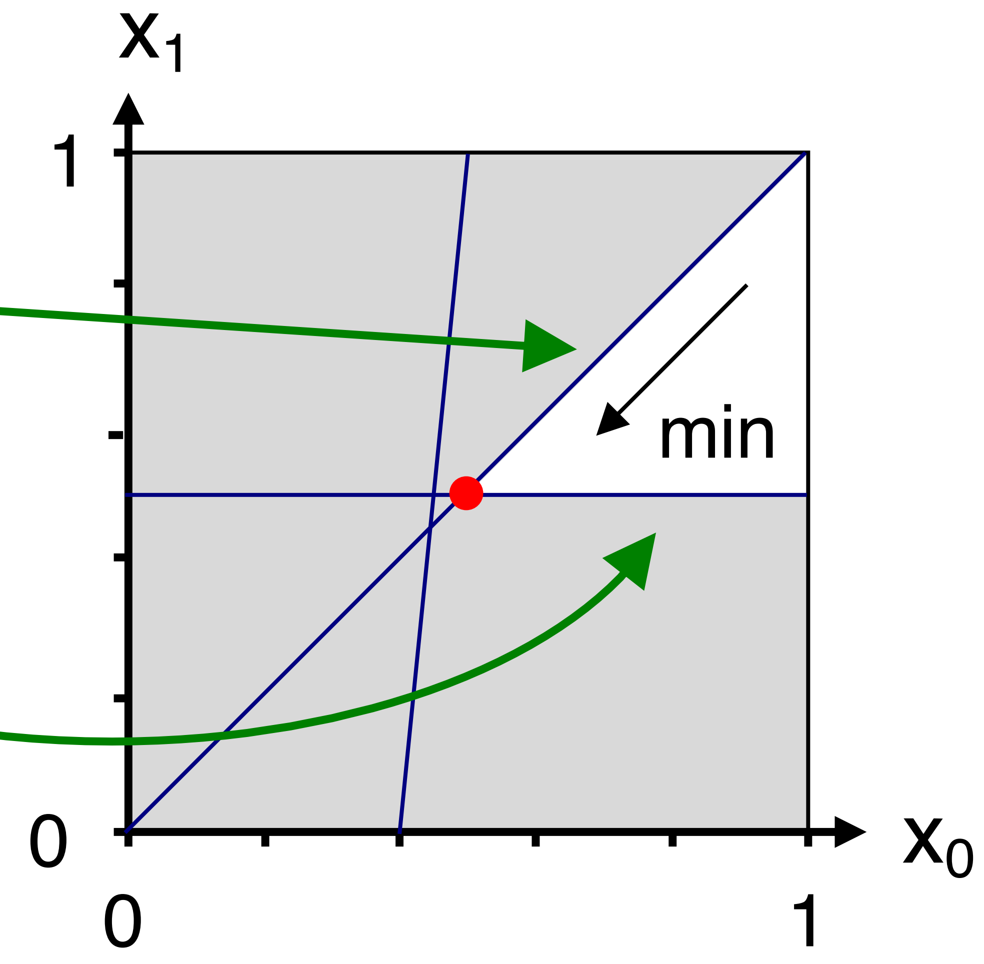


Minimise x_0+x_1 s.t.:
$x_0 \geq 0.4x_0 + 0.6x_1$
$x_0 \geq 0.1x_1 + 0.5x_3 + 0.4x_4$
$x_1 \geq x_2$
$x_1 \geq 0.5x_2 + 0.5x_4$

Minimise x_0+x_1 s.t.:
$x_0 \geq x_1$
$x_0 \geq 0.1x_1 + 0.4$
$x_1 \geq 0.5$

$x_0 \geq x_1$
(s_0:south)

$x_1 \geq 0.5$
(s_1:east)



Solving SSP for MDPs

- Value iteration:

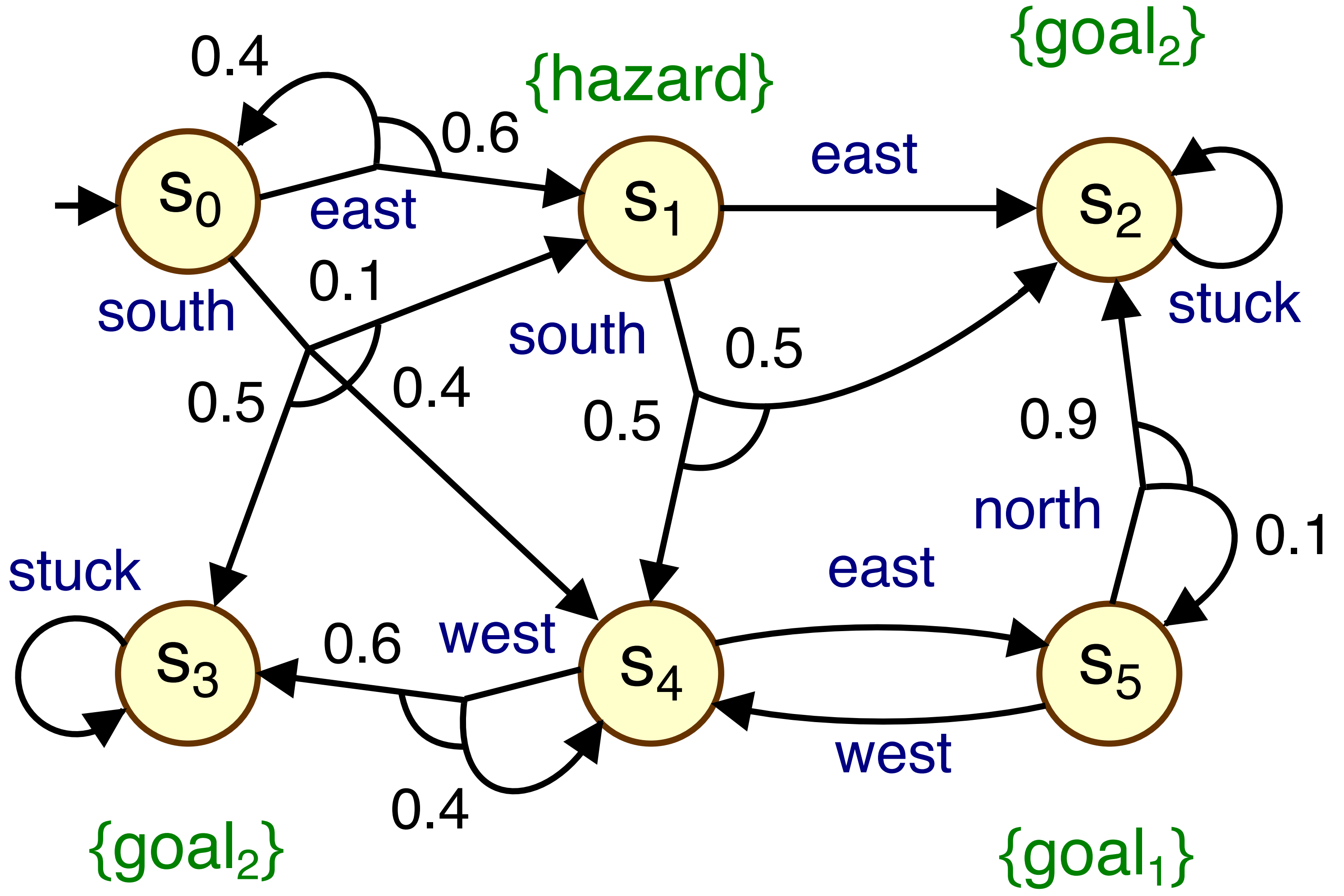
$$x_s^k = \begin{cases} 0 & \text{if } s \in goal \\ \min_{a \in A(s)} \left[C(s,a) + \sum_{s' \in S} P_s^a(s') \cdot x_{s'}^{k-1} \right] & \text{otherwise} \end{cases}$$

- Linear programming

maximise $\Sigma_{s \in S}\, x_s$ subject to the constraints:

$x_s = 0$ for $s \in goal$

$x_s \leq C(s,a) + \Sigma_{s' \in S} P_s^a(s') \cdot x_{s'}$ for $s \in S_?,\ a \in A(s)$



- Pre-computation:
  - we can also use graph-based pre-computation to identify/collapse states and relax SSP assumptions

MDP solution methods

- Solving MaxProb (or SSP) on MDPs (focusing on “exact” algorithms):

- Value iteration (VI)
  - simple, and effective in practice, but care needed with convergence detection
  - complexity unclear (depends on accuracy)

- Linear programming
  - polynomial complexity
  - in principle, can yield exact (arbitrary precision) optimal values; likely scales worse than VI

- Various other algorithms / optimisations
  - Policy iteration, VI + prioritisation, topological partitioning, parallelisation, …
  - Heuristics (e.g., BRTDP), sampling (e.g., Monte Carlo tree search), …

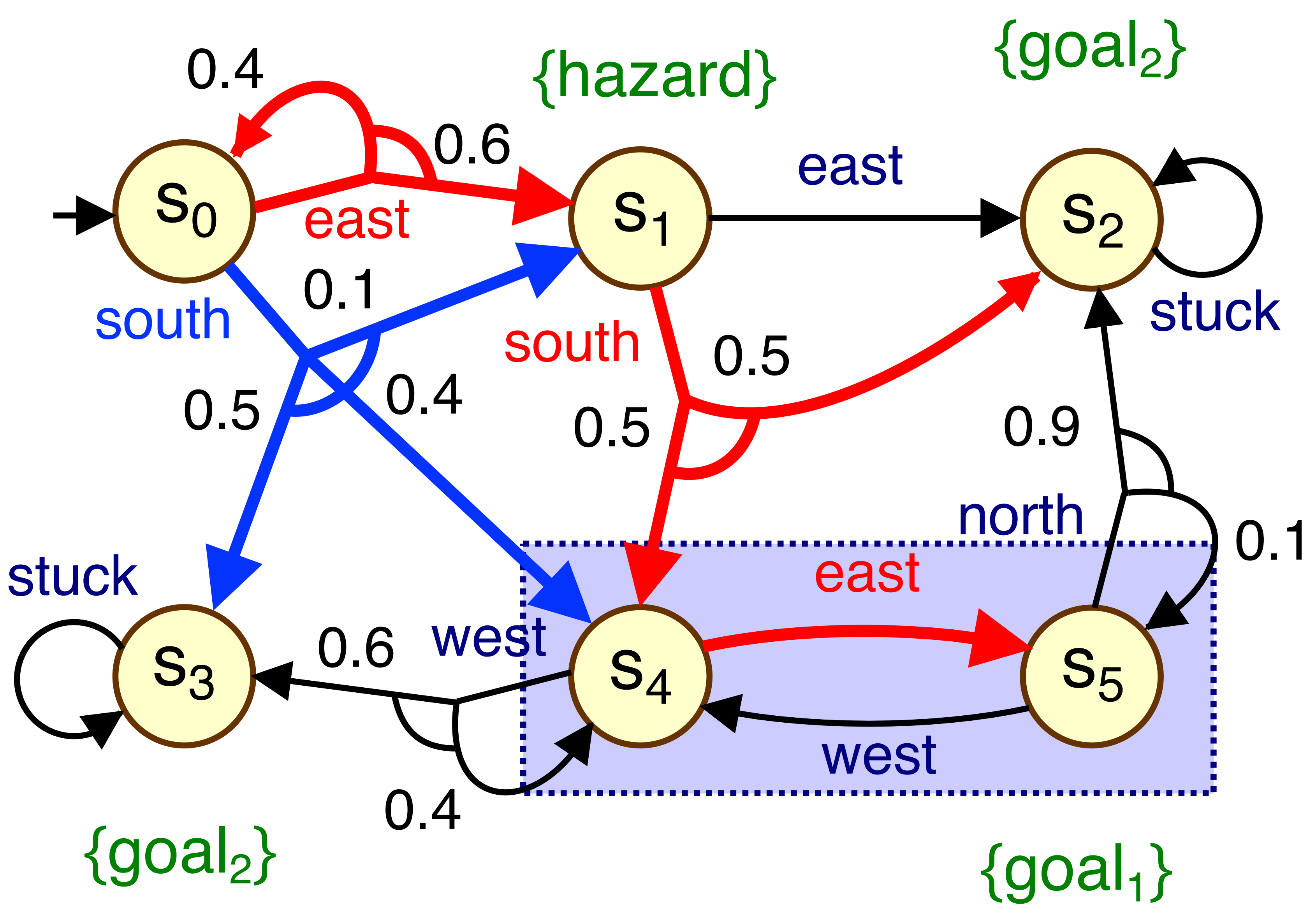
MaxProb over a finite horizon

- Finite-horizon variant solvable with value iteration (without pre-computation)
  - $V^*(s) = x_s^k$ where:

$$x_s^k = \begin{cases} 1 & \text{if } s \in goal \\ 0 & \text{if } s \notin goal \text{ and } n = 0 \\ \max_{a \in A(s)} \sum_{s' \in S} P_s^a(s') \cdot x_{s'}^{k-1} & \text{otherwise} \end{cases}$$

- Running example
  - $\text{MaxProb}^{\leq k}(\{s_4, s_5\})$
  - optimal policy is not memoryless

k	x_0	x_1
0	0	0
1	0.4	0.5
2	0.46	0.5
3	0.484	0.5



Beyond MDPs

- How do we go beyond the assumptions made so far?
- Full observability (of state, costs, …)
  - partially observable MDPs, beliefs over hidden state
- Finite state spaces, action spaces
  - continuous state/action, dynamic systems
- Full knowledge of the model
  - epistemic uncertainty, also sampling-based models
- Fully controllable model
  - adversarial (or collaborative) scenarios: stochastic game models

Summary (part 1)

- Markov decision processes
  - sequential decision making under (aleatoric) uncertainty
  - policies and objectives (MaxProb, SSP, finite-horizon, temporal logic)
  - solving MDPs (optimal policy generation)
    - linear programming (PTIME)
    - dynamic programming (value iteration)

- Next: Stochastic games (adding adversarial aspects)

- Next: Uncertain MDPs (adding epistemic uncertainty)

References (part 1)

- Applications & challenges
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